Journal of Statistical Physics, Vol. 38, Nos. 3/4, 1985

A Comparison Between Transitions Induced by Random and Periodic Fluctuations

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Received December 12, 1983; revised June 18, 1984

It is shown that a certain class of nonlinear systems possesses a unique and stable stationary state when subjected to periodic dichotomous modulations of an external parameter. This result enables us to define a probability density for the system and to characterize its shape and support. We compare this probability density with the one obtained in the case that the external parameter fluctuates randomly like a Markovian dichotomous noise and discuss various fluctuation-induced transition phenomena. The effects of these two types of fluctuations are quite dissimilar: the random fluctuations give rise to a richer behavior. The results are applied to the Freedericksz transition in nematic liquid crystals.

KEY WORDS: Dichotomous noise; periodic dichotomous modulations; fluctuation-induced transitions; Freedericksz transition.

1. INTRODUCTION

The state of a nonlinear system far from thermodynamic equilibrium strongly depends on the external constraints which the surroundings impose on the system. Most studies of the behavior of nonequilibrium systems and of the various transition phenomena which can occur in such systems have assumed that the environmental conditions show no temporal variations; see, for instance, Refs. 1 and 2. While this is a convenient assumption for theoretical and experimental studies, it nevertheless corresponds to an idealization. Environments of natural nonequilibrium systems often display a great amount of variability. Variations in the external constraints can be

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0022-4715/85/0200-0763$04.50/0 © 1985 Plenum Publishing Corporation

broken down essentially into two types: (i) periodic changes in the environment, and (ii) random environmental fluctuations. The effect of periodic perturbations has mainly been studied in the context of nonlinear oscillators. This is of special interest in biological applications, where the rhythm of autonomous biological oscillators can be markedly affected by such perturbations. (For recent reviews, see Refs. 3 and 4.) The second type of environmental variability has received increasing interest over the last few years. Here most studies consider one-variable systems which obey a first-order temporal evolution equation. (A few papers deal with two-variable systems.[5,6]) The evolution equation contains the influence of the environment in the form of some external parameter. Random environmental fluctuations are therefore modeled by replacing this parameter by a stationary stochastic process with a fixed mean value and a specified correlation function. The evolution equation of the system in question becomes a stochastic differential equation and the state of the system is given by a random variable. The latter is characterized by a probability distribution. It has been found, theoretically and experimentally, that external noise can give rise to unexpected transition phenomena: The state of the system, as characterized by its probability distribution, might change qualitatively while the variance or correlation time of the external noise is varied and the average value of the external constraints remains fixed.[7–11] For instance, the stationary probability density of the system might change from monomodal to bimodal behavior, even if the system does not display any bistable behavior for constant ambient conditions.[12,13] This can be called a noise-induced transition to bistability. In fact, as in the case of equilibrium phase transitions the extrema of the stationary probability density should be identified with the "phases," i.e., the macroscopic steady states of the system. The maxima correspond to stable steady states, the minima to unstable states. The main arguments for this identification are (i) the maxima are the most probable states and are therefore preferentially observed; and (ii) the extrema (and not some moments) converge, as the noise is "turned off," to the steady states of the system for constant external constraints. Thus the major upshot of studies on the effect of external noise is the existence of a new class of nonequilibrium transitions. Surprisingly, external noise can "create" new macroscopic states. At well-defined threshold values of the variance or correlation time of the noise, the macroscopic behavior of the system changes qualitatively, i.e., the system undergoes a nonequilibrium transition, a so-called noise-induced transition.

It is natural to ask if the observed noise-induced effects depend on the nature of the fluctuations of the environment. Do the same phenomena appear when the external parameter varies in a similar, but regular way? This is a multifaceted question. We wish to consider random and regular modulations which are qualitatively similar, and, in order to compare the effects of the different variations it is necessary to establish a well-defined stationary probability density for systems under the influence of nonrandom variations. Although studies and comparison of the moments of some systems subjected to random and periodic fluctuations have been made,[14] we adopt the stationary probability density as the criteria for comparison of systems under the different influences. We will see that the effects of the two types of variations in the systems we study are often quite different, in distinction from the conclusions of previous studies.

In this paper we restrict ourselves to specific types of random fluctuations for which we may define qualitatively similar types of periodic variations and ensure the existence of a stationary probability density for systems under its influence. The random fluctuations are of the form of the symmetric dichotomous Markov process, a process which takes on only two values with an exponentially distributed waiting time for the switches between the two values. This process is characterized by two parameters: the amplitude of the variation and the average switching frequency. Clearly, the analogous type of nonrandom modulation is one which switches between the same two values at a fixed frequency equal to the averge switching frequency of the random process.

Conveniently, the problem of determining the stationary probability density of a system under the influence of dichotomous Markov noise is an exactly soluble one. In Section 2 of this paper we review the properties of this process and the stationary density of the state variables of systems subjected to it. Section 3 is concerned with the periodic variations: we prove the existence of a stationary state and define a stationary probability density for a broad class of systems under the influence of such variations. A comparison of the effects induced by the two kinds of fluctuations is made in Section 4, and Section 5 contains a summary of our results and a discusion of the similarities and differences between the effects of random and periodic fluctuations.

2. THE RANDOM FLUCTUATIONS

The symmetric dichotomous Markov process, I_t, is a random process whose value switches between two values $\pm\Delta$ at random times. The waiting times in each state are exponentially distributed, which ensures the Markov character of the process. If the process starts with equal probabilities for each state it is a stationary process with mean value zero, $E\{I_t\} = 0$, and an exponentially decreasing correlation function

$$E\{I_t I_{t+\tau}\} = \Delta^2 e^{-\gamma\tau} \tag{1}$$

The correlation time is

$$\tau_{\text{cor}} = \gamma^{-1} \tag{2}$$

while the average frequency of transition from one state to the other is $\gamma/2$.

Systems which obey the deterministic evolution equation

$$\dot{x} = F(x, \lambda) \tag{3}$$

(where λ is the external parameter characterizing the environment) will, under the influence of fluctuations of this form, obey the stochastic differential equation

$$\dot{X}_t = F(X_t, \bar{\lambda} + I_t) \tag{4}$$

where $\bar{\lambda}$ is the mean value of λ. The stationary probability density of X_t may be evaluated exactly and is given by[15]

$$\begin{aligned} p_s(x) = N &\left\{\frac{1}{F(x, \bar{\lambda} + \Delta)} - \frac{1}{F(x, \bar{\lambda} - \Delta)}\right\} \\ &\times \exp\left\{-\frac{\gamma}{2}\int^x dx' \left[\frac{1}{F(x', \bar{\lambda} + \Delta)} + \frac{1}{F(x', \bar{\lambda} - \Delta)}\right]\right\} \end{aligned} \tag{5}$$

for x in the interior of the support $U = [x_-, x_+]$;

$$p_s(x) \equiv 0, \qquad x \notin U \tag{6}$$

and N is a normalization constant. The boundaries $x_\pm$ of the support obey the equation

$$0 = F(x_\pm, \bar{\lambda} \pm \Delta) \tag{7}$$

The significance of the support U is illustrated in Fig. 1. The extrema of $p_s(x)$ within the support are the points where the derivative vanishes, i.e., the solutions of

$$\begin{aligned} 0 = &-\left[F'(x, \bar{\lambda} + \Delta) + \frac{\gamma}{2}\right] F^2(x, \bar{\lambda} - \Delta) \\ &+ \left[F'(x, \bar{\lambda} - \Delta) + \frac{\gamma}{2}\right] F^2(x, \bar{\lambda} + \Delta) \end{aligned} \tag{8}$$

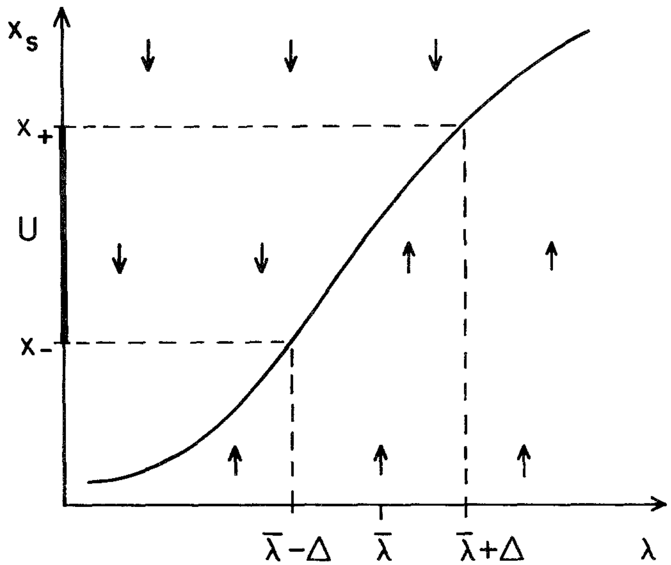


Fig. 1. A plot of the steady state x_s of $F(x, \lambda)$ vs. λ to illustrate the significance of the supported U. U is the region of state space between the steady states determined by the two values taken by the external parameter. The arrows indicate the direction of $\dot{x}$ as given by the deterministic evolution equation. Under the influence of dichotomous noise, the values of x will be restricted to U as $t \to \infty$.

The dichotomous Markov process has a white noise limit: if $\gamma \to \infty$ and $\Delta \to \infty$ such that

$$\frac{\Delta^2}{\gamma} = \frac{\sigma^2}{2} \qquad \text{(constant)} \tag{9}$$

I_t has the same effect on a system in which it appears linearly as does the Gaussian white noise process ξ_t[4] with amplitude σ.[15] Indeed, for evolution equations of the form

$$\dot{X}_t = f(X_t) + I_t h(X_t) \tag{10}$$

the stationary probability density for X_t, in the white noise limit, goes over to

$$p_{s_{\text{white}}}(x) = \tilde{N} h^{-1}(x) \exp\left\{\frac{2}{\sigma^2}\int^x dx' \, \frac{f(x')}{h^2(x')}\right\} \tag{11}$$

[4] With $E\{\xi_t\} = 0$, $E\{\xi_t \xi_{t+\tau}\} = \delta(\tau)$.

This is the stationary probability density of the process defined by the stochastic differential equation

$$\dot{X}_t = f(X_t) + \sigma h(X_t)\,\xi_t \tag{12}$$

in the Stratonovich interpretation.

3. THE PERIODIC FLUCTUATIONS

We now consider periodic fluctuations in the environment analogous to the random process treated in the previous section. As before, the external parameter switches between two values, $\bar{\lambda} \pm \Delta$. The switching, however, now occurs at definite time intervals of length $T/2$ (so that the period of the external parameter is T). The periodic variation will mimic the dichotomous Markov process with correlation time γ^{-1} when

$$\frac{2}{T} = \frac{\gamma}{2} \tag{13}$$

i.e., when the average switching frequencies of the two variations are the same.

Before we may define a stationary density for systems under the influence of these periodic fluctuations, we must establish the existence, uniqueness, and asymptotic stability of a stationary state. The stationary state is that in which the system returns to the same position after each cycle of the external parameter. The uniqueness and asymptotic stability of such a state then imply that the system will always approach this state independent of the initial condition. Once in this state, a probability density for the state variable may be defined in a straightforward manner.

The stationary state for a system described by the evolution equation

$$\dot{x} = F(x, \lambda(t)) \tag{14}$$

where

$$\lambda(t) = \begin{cases} \bar{\lambda} + \Delta, & nT < t < (n + 1/2)T, \qquad n = 0, 1, 2,... \\ \bar{\lambda} - \Delta, & (n + 1/2)T < t < (n + 1)T \end{cases} \tag{15}$$

is the solution $y(t)$ with the initial condition y_- which satisfies

$$y(nT) = y_-, \qquad n = 0, 1, 2,... \tag{16}$$

There is then also a value y_+ which satisfies

$$y((n + 1/2)T) = y_+, \qquad n = 0, 1, 2,... \tag{17}$$

and the two values $y_\pm$ are the solution to the simultaneous equations

$$\begin{aligned} \frac{T}{2} &= \int_{y_-}^{y_+} \frac{dx}{F(x, \bar{\lambda} + \Delta)} \\ \frac{T}{2} &= \int_{y_+}^{y_-} \frac{dx}{F(x, \bar{\lambda} - \Delta)} \end{aligned} \tag{18}$$

The existence of a stationary state is guaranteed by the existence of a solution to these simultaneous equations.

Likewise, the uniqueness of the stationary state is ensured by the uniqueness of the solution to (18). We establish the uniqueness of the stationary state for systems with a unique stable steady state for each value of the external parameter, i.e., $\bar{\lambda} - \Delta$ and $\bar{\lambda} + \Delta$, and with a unique stable steady state for the average force,

$$f(x, \bar{\lambda}, \Delta) \equiv \tfrac{1}{2}\{F(x, \bar{\lambda} + \Delta) + F(x, \bar{\lambda} - \Delta)\} \tag{19}$$

between the two stable steady states of $F(x, \bar{\lambda} \pm \Delta)$. To prove this, let us define

$$g(x, \bar{\lambda}, \Delta) \equiv \tfrac{1}{2}\{F(x, \bar{\lambda} + \Delta) - F(x, \bar{\lambda} - \Delta)\} \tag{20}$$

and $\bar{x}$ and $x_\pm$ (as before) by

$$0 = f(\bar{x}, \bar{\lambda}, \Delta), \qquad 0 = F(x_\pm, \bar{\lambda} \pm \Delta) \tag{21}$$

Without loss of generality we may take $x_+ > x_-$ so that

$$x_- < \bar{x} < x_+ \tag{22}$$

The stability of $x_\pm$ means that

$$F(x, \bar{\lambda} \pm \Delta) \lessgtr 0, \qquad x \in W, \qquad x \begin{cases} > x_+ \\ < x_- \end{cases} \tag{23}$$

where W is an interval contained in the intersection of the domains of $F(x, \bar{\lambda} \pm \Delta)$ with

$$[x_-, x_+] \subset W \tag{24}$$

such that

$$\begin{aligned} &F(x, \bar{\lambda} \pm \Delta) \gtrless 0, \qquad x \begin{cases} < x_- \\ > x_+ \end{cases} \\ &g(x, \bar{\lambda}, \Delta) > |f(x, \bar{\lambda}, \Delta)|, \qquad x \in (x_-, x_+) \end{aligned} \tag{25}$$

A graphical representation of the setup is given in Fig. 2.

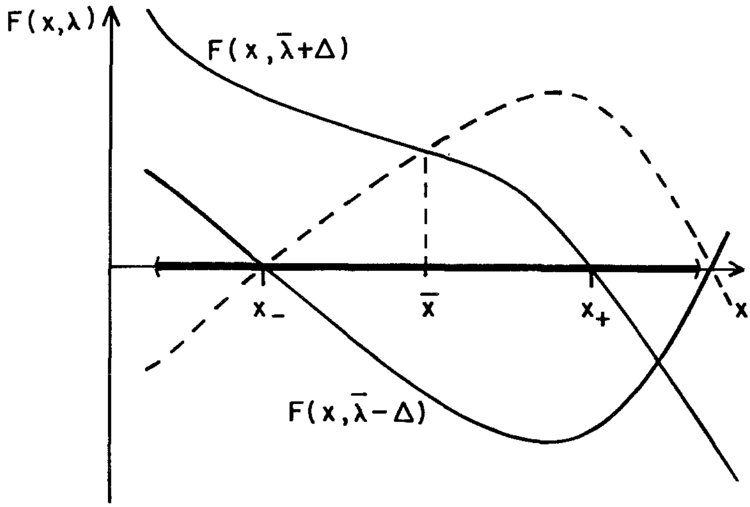


Fig. 2. An illustration of the setup for the proof of the uniqueness and stability of the stationary steady state. The forces $F(x, \bar{\lambda} \pm \Delta)$ each have one stable steady state, are continuous and bounded inside W. All solutions with initial conditions in W are attracted to the interval $[x_-, x_+]$. The broken line is $-F(x, \bar{\lambda} - \Delta)$.

The simultaneous equations (18) may be rewritten equivalently as

$$\begin{aligned} \frac{T}{2} &= \int_{y_-}^{y_+} dx \frac{g(x)}{g^2(x) - f^2(x)} = \mathscr{G}(y_+) - \mathscr{G}(y_-) \\ 0 &= \int_{y_-}^{y_+} dx \frac{f(x)}{g^2(x) - f^2(x)} = \mathscr{F}(y_+) - \mathscr{F}(y_-) \end{aligned} \tag{26}$$

where $\mathscr{G}(x)$ and $\mathscr{F}(x)$ are

$$\begin{aligned} \mathscr{G}(x) &= \int_{x_0}^{x} dx' \frac{g(x')}{g^2(x') - f^2(x')} \\ \mathscr{F}(x) &= \int_{x_0}^{x} dx' \frac{f(x')}{g^2(x') - f^2(x')} \end{aligned} \tag{27}$$

and $x_0 \in (x_-, x_+)$ is arbitrary. Clearly, any solution to the equations (18) must lie in the open set (x_-, x_+): this interval attracts all solutions with initial conditions in W. Equation (25) implies that

$$\mathscr{G}'(x) > 0, \qquad x \in (x_-, x_+) \tag{28}$$

so that $\mathscr{G}(x)$ is a monotonically increasing and hence invertible function on this set. We have required the stationary state of the average force to be stable, i.e.,

$$f(x, \bar{\lambda}, \Delta) \lesseqgtr 0, \qquad x \gtreqless \bar{x}, \qquad x \in (x_-, x_+) \tag{29}$$

Thus,

$$\mathscr{F}'(x) \lesseqgtr 0, \qquad x \gtreqless \bar{x}, \qquad x \in (x_-, x_+) \tag{30}$$

and we may define two monotonic functions with disjoint domains:

$$\begin{aligned} \mathscr{F}_+(x) &= \mathscr{F}(x) \qquad x \in (\bar{x}, x_+) \\ \mathscr{F}_-(x) &= \mathscr{F}(x) \qquad x \in (x_-, \bar{x}) \end{aligned} \tag{31}$$

Equations (26) may then be rewritten

$$y_+ = \mathscr{G}^{-1}(T/2 + \mathscr{G}(y_-)), \qquad y_+ = \mathscr{F}_+^{-1}(\mathscr{F}_-(y_-)) \tag{32}$$

Solutions of (26) correspond to the intersection in the (y_-, y_+) plane of the two curves given by (32). Since the first is monotonically increasing and the second is monotonically decreasing, they may intersect at most once. Hence any solution to (18) is unique.

The asymptotic stability of the stationary state follows from its uniqueness. To see this, consider the difference between any solution $x(t)$ and the stationary state $y(t)$ [with $y(0) = y_-$]

$$\delta(t) \equiv x(t) - y(t) \tag{33}$$

The change in this difference over one period of the external parameter,

$$\Delta\delta \equiv \int_0^T \dot{\delta}(t)\, dt = \int_0^T F(y(t) + \delta(t), \lambda(t))\, dt \tag{34}$$

is a function of the initial condition

$$\delta_0 \equiv \delta(t = 0) \tag{35}$$

We may deduce the qualitative features of the dependence of $\Delta\delta$ on δ_0:

1. $\Delta\delta$ is a continuous function of δ_0 for $F(x, \bar{\lambda} \pm \Delta)$ continuous functions of x;
2. $\Delta\delta \neq 0$ for $\delta_0 \neq 0$ by the uniqueness of the stationary state;
4. $\Delta\delta \lessgtr 0$ for $\delta_0 \left\{ \begin{smallmatrix} > x_+ - y_- \\ < x_- - y_- \end{smallmatrix} \right.$ since the interval (x_-, x_+) is attracting; and
5. $|\Delta\delta| < |\delta_0|$ for $\delta_0 \neq 0$ since the paths $x(t)$ and $y(t)$ can never intersect by the uniqueness of the solutions to Eq. (14).

These deductions sufficiently determine the behavior of $\Delta\delta(\delta_0)$ as illustrated in Fig. 3. We may then construct the bounded monotone sequence

$$\delta((n+1)T) = \delta(nT) + \Delta\delta(\delta(nT)), \qquad n = 0, 1, 2,... \tag{36}$$

which clearly converges to zero. Since $F(x, \bar{\lambda} \pm \Delta)$ are bounded on W there exists a constant C such that

$$|\delta(t)| \leqslant C\,|\delta(nT)|, \qquad nT \leqslant t \leqslant (n+1)T \tag{37}$$

for initial conditions in W. Equations (36) and (37) then imply that

$$\lim_{t \to \infty} x(t) = y(t) \tag{38}$$

for all solutions $x(t)$ with initial conditions in W.

The uniqueness and stability of the stationary state may in fact be established for a somewhat larger class of systems with a slightly restricted range of initial conditions. These are systems where, for instance, x_- is an unstable stationary state for $F(x, \bar{\lambda} + \Delta)$. (Of course, the + and − may be

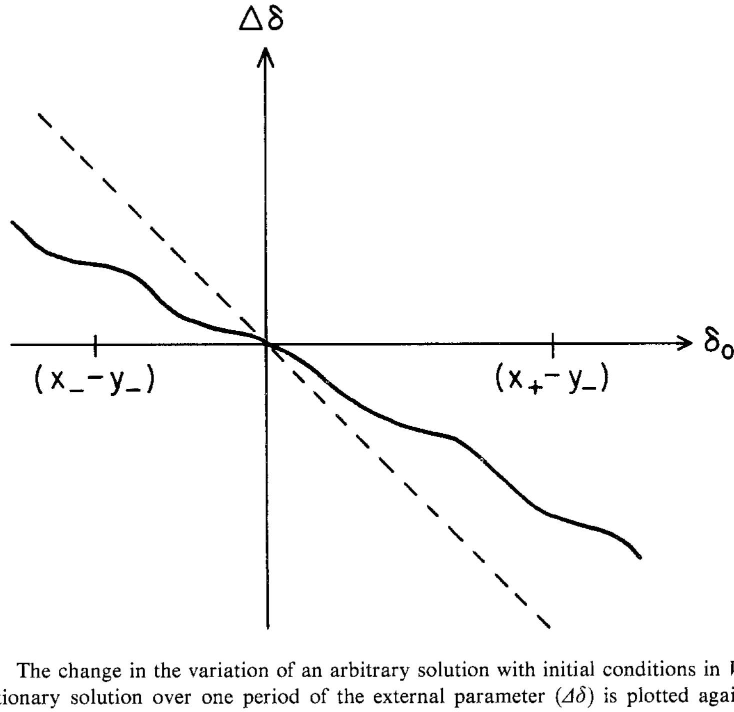


Fig. 3. The change in the variation of an arbitrary solution with initial conditions in W from the stationary solution over one period of the external parameter ($\Delta\delta$) is plotted against the initial variation (δ_0). The broken line is the curve $\Delta\delta = -\delta_0$.

exchanged or they may be considered simultaneously.) The proof of the uniqueness of the (nontrivial) stationary state goes as before with the restriction that all initial conditions must lie to the right of x_-. The proof of the stability of the stationary state is complicated by the fact that there is a trivial stationary state at x_-. In this case $\Delta\delta(\delta_0)$ vanishes at both $\delta_0 = 0$ and $\delta_0 = x_- - y_-$, and we cannot argue as before regarding the behavior of $\Delta\delta(\delta_0)$ for $\delta_0 < 0$. If, however, for some values of $\bar{\lambda}$ the curve $\Delta\delta(\delta_0)$ is of the form shown in Fig. 3 and the curve varies continuously with $\bar{\lambda}$, then it is clear that $\Delta\delta(\delta_0) > 0$ for $x_- - y_- < \delta_0 < 0$.

Once the system has relaxed to its stationary state, we may appeal to its ergodicity to find the stationary probability density. The density at a point x is proportional to the time spent in a neighborhood of x which is inversely proportional to the average of the absolute value of the velocity at x. For half the time the velocity is $F(x, \bar{\lambda} + \Delta)$ and the other half of the time the velocity is $F(x, \bar{\lambda} - \Delta)$. Hence, the stationary probability density may be written

$$q_s(x) = N \left\{ \frac{1}{|F(x, \bar{\lambda} + \Delta)|} + \frac{1}{|F(x, \bar{\lambda} - \Delta)|} \right\} \tag{39}$$

The support of $q_s(x)$ is the interval $V = [y_-, y_+] \subset U$ and given the previous assumptions on the nature of $F(x, \bar{\lambda} \pm \Delta)$ and Eq. (18) we have

$$q_s(x) = \nu \left\{ \frac{1}{F(x, \bar{\lambda} + \Delta)} - \frac{1}{F(x, \bar{\lambda} - \Delta)} \right\} \tag{40}$$

where

$$\nu = T^{-1} \tag{41}$$

and

$$q_s(x) \equiv 0, \qquad x \notin V \tag{42}$$

This density is bounded for all nonvanishing values of ν. The extrema of $q_s(x)$ inside the support V are the solutions of

$$0 = -F'(x, \bar{\lambda} + \Delta) F^2(x, \bar{\lambda} - \Delta) + F'(x, \bar{\lambda} - \Delta) F^2(x, \bar{\lambda} + \Delta) \tag{43}$$

For linear occurrences of the fluctuation amplitude, the support V collapses in the "white noise" limit examined in the previous section. If

$$F(x, \bar{\lambda} \pm \Delta) = f(x) \pm \Delta g(x) \tag{44}$$

then Eq. (18) may be restated

$$\frac{1}{2} \frac{\Delta^2}{\nu} = \Delta \int_{y_\mp}^{y_\pm} dx \frac{1}{(1/\Delta) f(x) \pm g(x)} \tag{45}$$

Since we are considering systems with $g(x) > 0$, the only way (45) can be satisfied when $\nu \to \infty$ and $\Delta \to \infty$ such that Δ^2/ν remains constant is for y_- and y_+ to coalesce.

4. RANDOM VS. PERIODIC FLUCTUATIONS

Note the similarity of the functional forms of the stationary density for the periodic fluctuations

$$q_s(x) = \nu \left\{ \frac{1}{F(x, \bar{\lambda} + \Delta)} - \frac{1}{F(x, \bar{\lambda} - \Delta)} \right\}, \qquad x \in V \tag{46}$$

and the stationary density for random fluctuations

$$p_s(x) = N \left\{ \frac{1}{F(x, \bar{\lambda} + \Delta)} - \frac{1}{F(x, \bar{\lambda} - \Delta)} \right\} \times \exp \left\{ -\frac{\gamma}{2} \int^x dx' \left[\frac{1}{F(x', \bar{\lambda} + \Delta)} + \frac{1}{F(x', \bar{\lambda} - \Delta)} \right] \right\}, \qquad x \in U \tag{47}$$

The reason for this similarity becomes apparent in the limit $\gamma = 4\nu \to 0$ where $V \to U$ and the exponential term in Eq. (47) becomes essentially unity. In this case, the fluctuations are occurring on a time scale much longer than the characteristic response time of the system. The response time, τ_{sys}, can be defined here as a typical decay time of the system near a deterministic steady state. For each value of the fluctuating parameter, the system almost completely relaxes to its deterministic value $x_\pm$. With the random variations, the probability that the parameter will switch before the system "achieves" its deterministic value is very small ($\gamma\tau_{\text{sys}} \ll 1$). Over a long period of time, the state variable will spend most of the time near the deterministic steady states whether the external parameter varies randomly or regularly. The distributions begin to differ for $\tau_{\text{corr}} \backsim T \backsim \tau_{\text{sys}}$ since the fluctuations are then occurring often enough for the system to "remember" the occurrences of previous switches.

The simplest systems available for the study of this difference are those with additive noise. In these cases, the force in the evolution equation can be written

$$F(x, \bar{\lambda} \pm \Delta) = f(x, \bar{\lambda}) \pm \Delta \tag{48}$$

For the random fluctuations there are several phenomena which may occur. The stationary density diverges at the boundaries of the support U for very long correlation times. As the correlation time decreases, so-called "hard" noise induced transitions occur at the correlation times where $p_s(x)$ ceases to diverge, but instead vanishes at one, and then the other boundary. In addition, the extrema of the density within the support U are the solutions of

$$0 = [f'(x, \bar{\lambda}) + \gamma/2] f(x, \bar{\lambda}) \tag{49}$$

The deterministic steady state [given by $f(x, \lambda) = 0$] is always an extremum, but there may be others in the support given by the solutions of

$$0 = f'(x, \bar{\lambda}) + \gamma/2 \tag{50}$$

For deterministically stable systems, $f'(x, \lambda)$ is often negative and of the order of the characteristic time of the system. We thus have the situation that the number of extrema inside the support may change for $\tau_{\text{corr}} \backsim \tau_{\text{sys}}$. This "soft" noise induced transition occurs even though there is no deterministic transition. (As a corollary, note that in the white noise limit in which $\gamma \to \infty$ there can be no such noise induced transitions with *additive* noise.)

If the same system is subjected to periodic fluctuations, then the stationary density $q_s(x)$ is bounded, positive definite and has extrema within the support V given by the solutions of

$$0 = f'(x, \bar{\lambda}) f(x, \bar{\lambda}) \tag{51}$$

The deterministic steady state is always an extremum, but there may be others if the derivative of the force vanishes inside the support V. Contrary to the case of random fluctuations the number of extrema can change only if the frequency and amplitude of the fluctuations are varied in such a way that a boundary of the support crosses an extremum. There are neither the soft nor the hard transitions that are possible with the random fluctuations although there may be these other fluctuation induced transitions. We find already in the simplest of systems effects which depend on the nature of the external fluctuations.

In more complicated systems where the external parameter occurs multiplicatively and nonlinearly, there is a complex interplay of the dependence of the support V of $q_s(x)$ on the frequency and amplitude, and of the dependence of the shape of $q_s(x)$ on the amplitude. There is a different complex interaction of the amplitude and correlation time which determine $p_s(x)$. It is convenient at this point to introduce a specific example to illustrate several of the various phenomena which can occur with the two types of fluctuations.

The physics of liquid crystals supplies us with a system in which the external field acts multiplicatively and nonlinearly in the evolution equation of the state variable. The Freedericksz instability in nematic liquid crystals has been studied extensively in the deterministic situation.[16] If a sample of nematic liquid crystal is layered between two properly prepared plates, the director (i.e., the direction vector of the molecular alignment) becomes constant in a direction determined by the boundaries. Because of the high anisotropy of the magnetic susceptibility, an externally applied magnetic field tends to force the director away from this constant configuration. The Freedericksz transition results from the competition between the boundary effects and the external field. The state of the system is given by the amplitude of the inhomogeneity of the director. In terms of normalized variables the deterministic evolution equation may be written

$$\dot{x} = \tau^{-1} x(\lambda^2 - 1 - \tfrac{1}{2}\lambda^2 x^2) \tag{52}$$

where τ is the zero field ($\lambda = 0$) relaxation time. There is a deterministic transition at $\lambda = 1$: for $\lambda < 1$, $x = 0$ is the unique stable steady state, while for $\lambda > 1$, $x = 0$ becomes unstable and the states

$$x = \pm[2(1 - \lambda^{-2})]^{1/2} \tag{53}$$

become stable. A complete analysis of the behavior of this system when the external parameter varies from its average value by the symmetric dichotomous Markov process has been carried out in Ref. 17. The results are neatly summarized by the phase diagrams in Fig. 4. Here the qualitative behavior of the stationary probability density is given as a function of the relative rate of the fluctuations $\gamma\tau$ and the amplitude Δ for several average values of the external field $\bar{\lambda}$. Both hard and soft noise induced transitions occur in this example.

For a periodically varying external field the stationary probability density for the state variable is

$$q_s(x) = \frac{\nu\tau}{x} \left\{ \frac{1}{(\bar{\lambda} + \Delta)^2 - 1 - \frac{1}{2}(\bar{\lambda} + \Delta)^2 x^2} - \frac{1}{(\bar{\lambda} - \Delta)^2 - 1 - \frac{1}{2}(\bar{\lambda} - \Delta)^2 x^2} \right\} \tag{54}$$

for x in the support $V = [y_-, y_+]$ where

$$y_\pm^2 = \frac{z_\pm z_\mp(\sigma_\pm \sigma_\mp - 1)}{z_\pm(\sigma_\mp - 1) + z_\pm \sigma_\mp(\sigma_\pm - 1)} \tag{55}$$

$$z_\pm = 2[1 - (\bar{\lambda} \pm \Delta)^{-2}] \tag{56}$$

$$\sigma_\pm = \exp\left[\frac{(\bar{\lambda} \pm \Delta)^2 - 1}{2\nu\tau}\right] \tag{57}$$

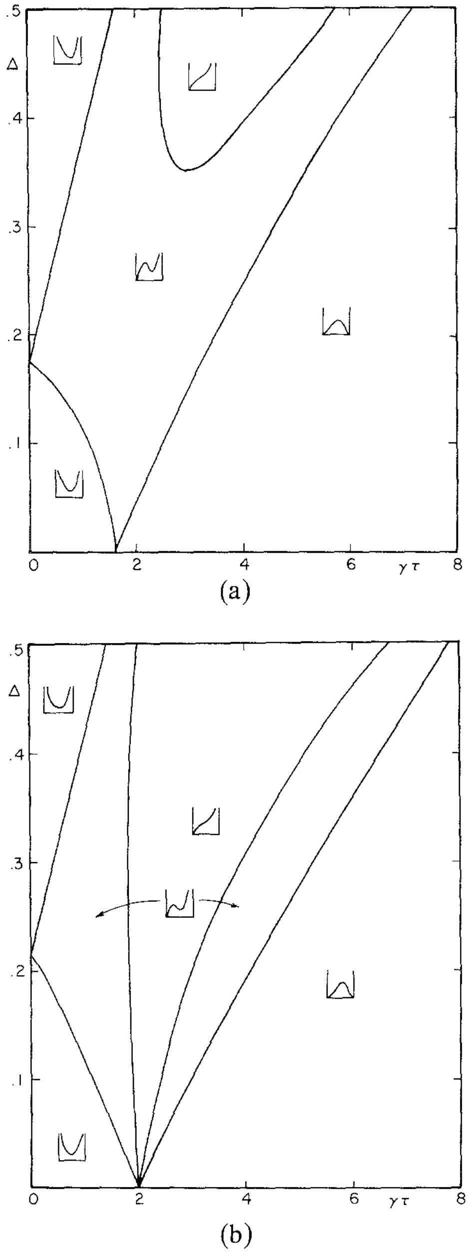


Fig. 4. Phase diagrams for the system described by Eq. (52) under the influence of dichotomous Markov noise in the $\gamma\tau$–Δ plane for two values of the average external field: (a) $\lambda = 1.175$, (b) $\lambda = 1.225$. The shape of the probability density has been sketched in each region.

For $\bar{\lambda} + \Delta < 1$ the density is concentrated at $x = 0$ while for $\bar{\lambda} + \Delta > 1$ a true density exists. Explicit calculation shows that there is only one extremum (a minimum) of the density $q_s(x)$ for $x \in U \supset V$ in the region of physically relevant $\bar{\lambda}$ and Δ. Hence in this example there are no soft fluctuation induced transitions such as occur with the random variations.

The boundaries of the support can vary, however, in such a way that the distribution changes from having two maxima on the boundaries to a monotonic shape. In fact, the direction of the monotonicity can vary with the amplitude of the variations. Consider the general situation for the moment. As $\nu \to \infty$, the boundaries of V coalesce to the same value $\bar{x}$, the stationary state of the average force. The transitions described above occur if

$$q_s'(\bar{x}) \neq 0 \tag{58}$$

The sign of the derivative of the stationary density at $\bar{x}$ is thus an indicator of the shape of the density at high frequencies.

We can construct a phase diagram in the (ν–Δ) plane showing the behavior of $q_s(x)$ as follows:

1. Find $\bar{x}$ as a function of Δ.
2. Find the values of Δ where $q_s'(\bar{x}) = 0$.
3. For the values of Δ in each interval where $q_s'(\bar{x}(\Delta)) > 0$ [<0] find the curve $\nu(\Delta)$ determined by $q_s'(y_-(\nu, \Delta)) = 0$ [$q_s'(y_+(\nu, \Delta)) = 0$].

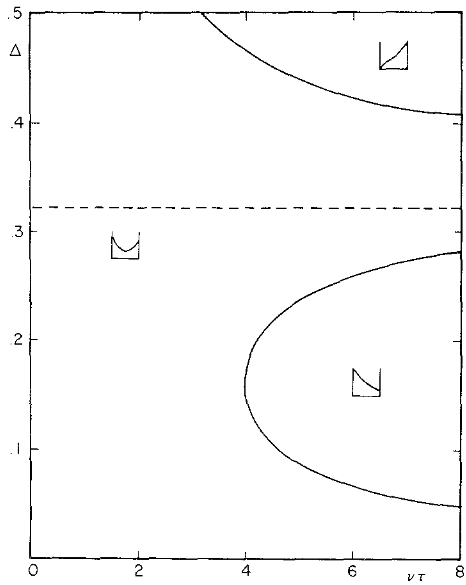
When $q_s'(y_-) = 0$, for instance, the density changes between a monomodal and a monotone increasing behavior. The curves $\nu(\Delta)$ give a partitioning of the ν–Δ plane each region of which corresponds to one qualitative state of the system. [In the above we have assumed that $q_s(x)$ has only one extremum in U for each Δ. If this is not the case the transition structure is more complicated but the construction of the phase diagram is easily generalized.] For the example under consideration,

$$\bar{x} = [2(1 - (\bar{\lambda}^2 + \Delta^2)^{-1})]^{1/2} \tag{59}$$

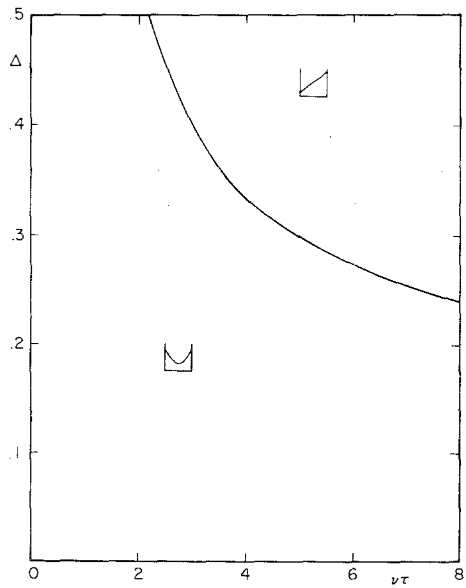
and the "critical" values of the fluctuation amplitude are

$$\Delta = 0, \qquad \Delta = (3/2 - \bar{\lambda}^2)^{1/2} \tag{60}$$

Figure 5 shows the phase diagrams constructed for several values of $\bar{\lambda}$. In Fig. 5a the probability density of the system displays transitions to both monotone increasing and monotone decreasing behaviors from the monomodal state observed for low frequencies. When the amplitude of the fluctuations is less than the critical value, the probability density has a peak at the lower end of its support for high frequencies. Above the critical



(a)



(b)

Fig. 5. Phase diagrams for the system described by Eq. (52) under the influence of periodic fluctuations in the $\nu\tau$–Δ plane for two values of the average external field: (a) $\lambda = 1.175$ and (b) $\lambda = 1.225$. The shape of the stationary density is sketched in each region. The broken line in (a) is at the "critical" value of the fluctuation amplitude [Eq. (60)].

amplitude, the transition is to a state where the peak of the density coincides with the upper end of the support. When $\bar{\lambda}^2 > 3/2$ as in Fig. 5b, there is only the trivial critical amplitude and for all amplitudes of the modulations the density has a peak at the upper end of the support for high frequencies. The comparison of Figs. 4 and 5 shows the marked dissimilarity in the qualitative behavior of this system under the influence of periodic and random noise when the time scale of the fluctuations is of the order of τ_{sys}.

Several features of the fluctuation induced transitions that occur in this example are due specifically to two characteristics of the system: the multiplicative nature and the nonlinear occurrence of the external parameter. For additive fluctuations $q_s'(\bar{x})$ vanishes identically and the stationary density will never change qualitatively at high frequencies. For fluctuations occurring linearly, $\bar{x}$ does not depend on Δ so that there are no (nontrivial) "critical" values of Δ. That is, in this situation there can be at most one type of transition to a monotone density (either to a monotonically increasing or decreasing density, but not both).

5. CONCLUSIONS

There are two main results of this paper. First, we have shown the uniqueness and stability of a stationary state for a broad class of systems subjected to periodic environmental fluctuations. This has allowed us to define a probability distribution for the state variables describing these systems and study the various fluctuation induced qualitative changes in the density. Secondly, a side-by-side comparison of the noise induced phenomena for these periodic and the similar random fluctuations has been made.

We are able to make definite statements about the effects that the two kinds of variations have on the behavior of systems as determined by their stationary probability densities. For very small or very large fluctuation rates (for fixed amplitudes), systems behave similarly under the influence of the random or periodic modulations. If the time scale of the noise is of the order of the characteristic response time of the system, various fluctuation induced transitions can occur but the effects of the two types of noise are quite distinct. This is not surprising. When a system is fast enough to respond significantly to a variation in its environment it is highly correlated with its surroundings. It is interesting that there is in general more transition structure when the fluctuations are of the form of the dichotomous Markov process than when they are regular. The appearance of the external parameter must become increasingly more complex (multiplicative, nonlinear) for the various transitions to occur with the periodic variations, whereas even more phenomena can occur in the simplest systems under the influence of the random fluctuations.

There can be random noise induced transitions for very fast fluctuations that occur linearly and multiplicatively if the amplitude is scaled accordingly. This can be seen by considering the number and location of the extrema of the density $p_s(x)$ in the white noise limit [see Eq. (11)]. In this same limit the support of the density $q_s(x)$ collapses. The difference between the supports of $p_s(x)$ and $q_s(x)$ gives us some insight into the nature of the mechanism responsible for the way a system discriminates between the two modulations at high fluctuation rates.

The spectral density of the dichotomous Markov process, i.e., the Fourier transform of the correlation function given by Eq. (1), is the Lorentzian

$$S_{\mathrm{ran}}(\omega) = \frac{\gamma \Delta^2}{\pi(\omega^2 + \gamma^2)} \tag{61}$$

The correlation function of the periodic fluctuations is a sawtooth wave between $\pm\Delta^2$ of fundamental (angular) frequency $2\pi\nu$ with Fourier transform

$$S_{\mathrm{per}}(\omega) = \sum_{n=\pm 1, \pm 3, \pm 5, \ldots} \frac{16\Delta^2\nu^2}{\omega^2}\, \delta(\omega - 2\pi n\nu) \tag{62}$$

When $\nu \ll \tau_{\mathrm{sys}}^{-1}$, the δ functions in Eq. (62) are very closely spaced and we may approximate the spectrum (62) by the histogram in Fig. 6. Also plotted in Fig. 6 is (61) for $\gamma = 4\nu$. The correspondence of the two spectral densities is consistent which the observed similarities in probability distributions of the system for slow fluctuations.

If $\nu \gg \tau_{\mathrm{sys}}^{-1}$, we are no longer justified in smoothing out the δ functions in (62). There are large voids in the spectral density of the periodic modulations, most significantly at the low frequencies. The presence of these low-frequency components in $S_{\mathrm{ran}}(\omega)$, to which the system may respond, is the agent preventing the collapse of the support of $p_s(x)$. In the white noise limit, where

$$S_{\mathrm{ran}}(\omega) \to \frac{\sigma^2}{2\pi} \tag{63}$$

there is an important low-frequency contribution to the power spectrum that is missing in the spectral distribution of the periodic fluctuations. A common characteristic of all genuinely random processes is the vanishing of the correlation function for long times and hence the continuity of the spectral

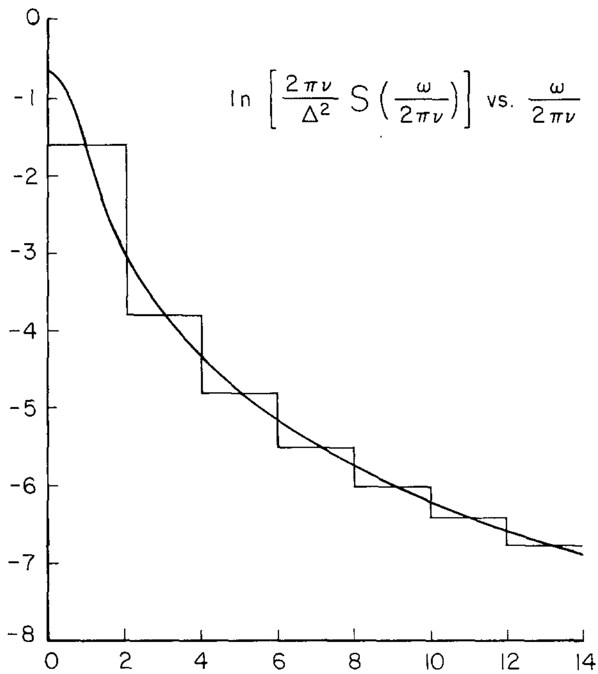


Fig. 6. The histogram is the (log) of the normalized spectral density of the periodic fluctuations overaged over intervals of length 2 on the $\omega/2\pi\nu$ axis. The solid line is the (log of the normalized) spectral density of the dichotomous Markov process for $\gamma = 4\nu$ [Eq. (61)].

density. No matter how complex the regular variations, any periodicity in the fluctuations will result in a discrete spectrum and the absence of key low-frequency components.

ACKNOWLEDGMENTS

We thank David Sigeti for a stimulating discussion and fruitful suggestions and Steven D. Kawaler for help with numerical computations. This work was partially supported by the U.S. Department of Energy, Office of Basic Energy Sciences, grant No. DE-AS05-81ER10947.

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